

K-THEORY OF THE LEAF SPACE OF FOLIATIONS FORMED BY THE GENERIC K-ORBITS OF SOME INDECOMPOSABLE MD_5 -GROUPS

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Abstract

The paper is a continuation of the authors' work [18]. In [18], we consider foliations formed by the maximal dimensional K-orbits (MD_5 -foliations) of connected MD_5 -groups such that their Lie algebras have 4-dimensional commutative derived ideals and give a topological classification of the considered foliations. In this paper, we study K-theory of the leaf space of some of these MD_5 -foliations and characterize the Connes' C*-algebras of the considered foliations by the method of K-functors.

INTRODUCTION

In the decades 1970s – 1980s, works of D.N. Diep [4], J. Rosenberg [10], G. G. Kasparov [7], V. M. Son and H. H. Viet [12],... have seen that K-functors are well adapted to characterize a large class of group C*-algebras. Kirillov's method of orbits allows to find out the class of Lie groups MD, for which the group C*-algebras can be characterized by means of suitable K-functors (see [5]). In terms of D. N. Diep, an MD-group of dimension n (for short, an MD_n -group) is an n -dimensional solvable real Lie group whose orbits in the co-adjoint representation (i.e., the K- representation) are the orbits of zero or maximal dimension. The Lie algebra of an MD_n -group is called an MD_n -algebra (see [5, Section 4.1]).

In 1982, studying foliated manifolds, A. Connes [3] introduced the notion of C*-algebra associated to a measured foliation. In the case of Reeb foliations (see A. M. Torpe [14]), the method of K-functors has been proved to be very effective in describing the structure of Connes' C*-algebras. For every MD-group G , the family of K-orbits of maximal dimension forms a measured foliation in terms of Connes [3]. This foliation is called MD-foliation associated to G .

⁰**Key words:** Lie group, Lie algebra, MD_5 -group, MD_5 -algebra, K-orbit, Foliation, Measured foliation, C*-algebra, Connes' C*-algebra associated to a measured foliation.

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Combining methods of Kirillov (see [8, Section 15]) and Connes (see [3, Section 2, 5]), the first author had studied MD_4 -foliations associated with all indecomposable connected MD_4 -groups and characterized Connes' C^* -algebras of these foliations in [16]. Recently, Vu and Shum [17] have classified, up to isomorphism, all the MD_5 -algebras having commutative derived ideals.

In [18], we have given a topological classification of MD_5 -foliations associated to the indecomposable connected and simply connected MD_5 -groups, such that MD_5 -algebras of them have 4-dimensional commutative derived ideals. There are exactly 3 topological types of the considered MD_5 -foliations, denoted by $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$. All MD_5 -foliations of type \mathcal{F}_1 are the trivial fibrations with connected fibre on 3-dimensional sphere S^3 , so Connes' C^* -algebras of them are isomorphic to the C^* -algebra $C(S^3) \otimes \mathcal{K}$ following [3, Section 5], where \mathcal{K} denotes the C^* -algebra of compact operators on an (infinite dimensional separable) Hilbert space.

The purpose of this paper is to study K-theory of the leaf space and to characterize the structure of Connes' C^* -algebras $C^*(V, \mathcal{F})$ of all MD_5 -foliations (V, \mathcal{F}) of type \mathcal{F}_2 by the method of K-functors. Namely, we will express $C^*(V, \mathcal{F})$ by two repeated extensions of the form

$$\begin{aligned} 0 &\longrightarrow C_0(X_1) \otimes \mathcal{K} \longrightarrow C^*(V, \mathcal{F}) \longrightarrow B_1 \longrightarrow 0, \\ 0 &\longrightarrow C_0(X_2) \otimes \mathcal{K} \longrightarrow B_1 \longrightarrow C_0(Y_2) \otimes \mathcal{K} \longrightarrow 0, \end{aligned}$$

then we will compute the invariant system of $C^*(V, \mathcal{F})$ with respect to these extensions. If the given C^* -algebras are isomorphic to the reduced crossed products of the form $C_0(V) \rtimes H$, where H is a Lie group, we can use the Thom-Connes isomorphism to compute the connecting map δ_0, δ_1 .

In another paper, we will study the similar problem for all MD_5 -foliations of type \mathcal{F}_3 .

1 THE MD_5 -FOLIATIONS OF TYPE \mathcal{F}_2

Originally, we will recall geometry of K-orbit of MD_5 -groups which associate with MD_5 -foliations of type \mathcal{F}_2 (see [18]).

In this section, G will be always an connected and simply connected MD_5 -group such that its Lie algebras \mathcal{G} is an indecomposable MD_5 -algebra generated by $\{X_1, X_2, X_3, X_4, X_5\}$ with $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \cong \mathbb{R}^4$, $ad_{X_1} \in End(\mathcal{G}) \equiv Mat_4(\mathbb{R})$. Namely, \mathcal{G} will be one of the following Lie algebras which are studied in [17] and [18].

1. $\mathcal{G}_{5,4,11}(\lambda_1, \lambda_1, \varphi)$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}; \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi).$$

2. $\mathcal{G}_{5,4,12(\lambda,\varphi)}$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

3. $\mathcal{G}_{5,4,13(\lambda,\varphi)}$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

The connected and simply connected Lie groups corresponding to these algebras are denoted by $G_{5,4,11(\lambda_1,\lambda_1,\varphi)}$, $G_{5,4,12(\lambda,\varphi)}$, $G_{5,4,13(\lambda,\varphi)}$. All of these Lie groups are MD_5 -groups (see [17]) and G is one of them. We now recall the geometric description of the K -orbits of G in the dual space \mathcal{G}^* of \mathcal{G} . Let $\{X_1^*, X_2^*, X_3^*, X_4^*, X_5^*\}$ be the basis in \mathcal{G}^* dual to the basis $\{X_1, X_2, X_3, X_4, X_5\}$ in \mathcal{G} . Denote by Ω_F the K -orbit of G including $F = (\alpha, \beta + i\gamma, \delta, \sigma)$ in $\mathcal{G}^* \cong \mathbb{R}^5$.

- If $\beta + i\gamma = \delta = \sigma = 0$ then $\Omega_F = \{F\}$ (the 0-dimensional orbit).
- If $|\beta + i\gamma|^2 + \delta^2 + \sigma^2 \neq 0$ then Ω_F is the 2-dimensional orbit as follows

$$\Omega_F = \begin{cases} \left\{ \left(x, (\beta + i\gamma) \cdot e^{(a \cdot e^{-i\varphi})}, \delta \cdot e^{a\lambda_1}, \sigma \cdot e^{a\lambda_2} \right), x, a \in \mathbb{R} \right\} \\ \quad \text{when } G = G_{5,4,11(\lambda_1,\lambda_2,\varphi)}, \lambda_1, \lambda_2 \in \mathbb{R}^*, \varphi \in (0; \pi). \\ \left\{ \left(x, (\beta + i\gamma) \cdot e^{(a \cdot e^{-i\varphi})}, \delta \cdot e^{a\lambda}, \sigma \cdot e^{a\lambda} \right), x, a \in \mathbb{R} \right\} \\ \quad \text{when } G = G_{5,4,12(\lambda,\varphi)}, \lambda \in \mathbb{R}^*, \varphi \in (0; \pi). \\ \left\{ \left(x, (\beta + i\gamma) \cdot e^{(a \cdot e^{-i\varphi})}, \delta \cdot e^{a\lambda}, \delta \cdot a e^{a\lambda} + \sigma \cdot e^{a\lambda} \right), x, a \in \mathbb{R} \right\} \\ \quad \text{when } G = G_{5,4,13(\lambda,\varphi)}, \lambda \in \mathbb{R}^*, \varphi \in (0; \pi). \end{cases}$$

In [18], we have shown that, the family \mathcal{F} of maximal-dimensional K -orbits of G forms measured foliation in terms of Connes on the open submanifold

$$V = \{(x, y, z, t, s) \in G^* : y^2 + z^2 + t^2 + s^2 \neq 0\} \cong \mathbb{R} \times (\mathbb{R}^4)^* (\subset \mathcal{G}^* \equiv \mathbb{R}^5)$$

Furthermore, all foliations $(V, \mathcal{F}_{4,11(\lambda_1,\lambda_2,\varphi)})$, $(V, \mathcal{F}_{4,12(\lambda,\varphi)})$, $(V, \mathcal{F}_{4,13(\lambda,\varphi)})$ are topologically equivalent to each other ($\lambda_1, \lambda_2, \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0; \pi)$). Thus, we need only choose a envoy among them to describe the structure of the C^* -algebra. In this case, we choose the foliation $(V, \mathcal{F}_{4,12(1, \frac{\pi}{2})})$.

In [18], we have described the foliation $(V, \mathcal{F}_{4,12(1, \frac{\pi}{2})})$ by a suitable action of \mathbb{R}^2 . Namely, we have the following proposition.

PROPOSITION 1. *The foliation $(V, \mathcal{F}_{4,12(1, \frac{\pi}{2})})$ can be given by an action of the commutative Lie group \mathbb{R}^2 on the manifold V .*

Proof. One needs only to verify that the following action λ of \mathbb{R}^2 on V gives the foliation $(V, \mathcal{F}_{4,12(1, \frac{\pi}{2})})$

$\lambda : \mathbb{R}^2 \times V \rightarrow V$

$((r, a), (x, y + iz, t, s)) \mapsto (x + r, (y + iz) \cdot e^{-ia}, t \cdot e^a, s \cdot e^a)$

where $(r, a) \in \mathbb{R}^2$, $(x, y + iz, t, s) \in V \cong \mathbb{R} \times (\mathbb{C} \times \mathbb{R}^2)^* \cong \mathbb{R} \times (\mathbb{R}^4)^*$. Hereafter, for simplicity of notation, we write (V, \mathcal{F}) instead of $(V, \mathcal{F}_{4,12(1, \frac{\pi}{2})})$. □

It is easy to see that the graph of (V, \mathcal{F}) is identical with $V \times \mathbb{R}^2$, so by [3, Section 5], it follows from Proposition 1 that:

COROLLARY 1 (analytical description of $C^*(V, \mathcal{F})$). *The Connes C^* -algebra $C^*(V, \mathcal{F})$ can be analytically described the reduced crossed of $C_0(V)$ by \mathbb{R}^2 as follows*

$$C^*(V, \mathcal{F}) \cong C_0(V) \rtimes_{\lambda} \mathbb{R}^2.$$
□

2 $C^*(V, \mathcal{F})$ AS TWO REPEATED EXTENSIONS

2.1. Let V_1, W_1, V_2, W_2 be the following submanifolds of V

$$\begin{aligned} V_1 &= \{(x, y, z, t, s) \in V : s \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^*, \\ W_1 &= V \setminus V_1 = \{(x, y, z, t, s) \in V : s = 0\} \cong \mathbb{R} \times (\mathbb{R}^3)^* \times \{0\} \cong \mathbb{R} \times (\mathbb{R}^3)^*, \\ V_2 &= \{(x, y, z, t, 0) \in W_1 : t \neq 0\} \cong \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*, \\ W_2 &= W_1 \setminus V_2 = \{(x, y, z, t, 0) \in W_1 : t = 0\} \cong \mathbb{R} \times (\mathbb{R}^2)^*. \end{aligned}$$

It is easy to see that the action λ in Proposition 1 preserves the subsets V_1, W_1, V_2, W_2 . Let i_1, i_2, μ_1, μ_2 be the inclusions and the restrictions

$$\begin{aligned} i_1 : C_0(V_1) &\rightarrow C_0(V), & i_2 : C_0(V_2) &\rightarrow C_0(W_1), \\ \mu_1 : C_0(V) &\rightarrow C_0(W_1), & \mu_2 : C_0(W_1) &\rightarrow C_0(W_2) \end{aligned}$$

where each function of $C_0(V_1)$ (resp. $C_0(V_2)$) is extended to the one of $C_0(V)$ (resp. $C_0(W_1)$) by taking the value of zero outside V_1 (resp. V_2).

It is known a fact that i_1, i_2, μ_1, μ_2 are λ -equivariant and the following sequences are equivariantly exact:

$$(2.1.1) \quad 0 \longrightarrow C_0(V_1) \xrightarrow{i_1} C_0(V) \xrightarrow{\mu_1} C_0(W_1) \longrightarrow 0$$

$$(2.1.2) \quad 0 \longrightarrow C_0(V_2) \xrightarrow{i_2} C_0(W_1) \xrightarrow{\mu_2} C_0(W_2) \longrightarrow 0.$$

2.2. Now we denote by $(V_1, \mathcal{F}_1), (W_1, \mathcal{F}_1), (V_2, \mathcal{F}_2), (W_2, \mathcal{F}_2)$ the foliations-restrictions of (V, \mathcal{F}) on V_1, W_1, V_2, W_2 respectively.

THEOREM 1. $C^*(V, \mathcal{F})$ admits the following canonical repeated extensions

$$(\gamma_1) \quad 0 \longrightarrow J_1 \xrightarrow{\widehat{i}_1} C^*(V, \mathcal{F}) \xrightarrow{\widehat{\mu}_1} B_1 \longrightarrow 0,$$

$$(\gamma_2) \quad 0 \longrightarrow J_2 \xrightarrow{\widehat{i}_2} B_1 \xrightarrow{\widehat{\mu}_2} B_2 \longrightarrow 0,$$

where

$$\begin{aligned} J_1 &= C^*(V_1, \mathcal{F}_1) \cong C_0(V_1) \rtimes_{\lambda} \mathbb{R}^2 \cong C_0(\mathbb{R}^3 \cup \mathbb{R}^3) \otimes K, \\ J_2 &= C^*(V_2, \mathcal{F}_2) \cong C_0(V_2) \rtimes_{\lambda} \mathbb{R}^2 \cong C_0(\mathbb{R}^2 \cup \mathbb{R}^2) \otimes K, \\ B_2 &= C^*(W_2, \mathcal{F}_2) \cong C_0(W_2) \rtimes_{\lambda} \mathbb{R}^2 \cong C_0(\mathbb{R}_+) \otimes K, \\ B_1 &= C^*(W_1, \mathcal{F}_1) \cong C_0(W_1) \rtimes_{\lambda} \mathbb{R}^2, \text{ and the homomorphisms } \widehat{i}_1, \widehat{i}_2, \widehat{\mu}_1, \widehat{\mu}_2 \text{ are defined by} \\ (\widehat{i}_k f)(r, s) &= i_k f(r, s), \quad k = 1, 2 \\ (\widehat{\mu}_k f)(r, s) &= \mu_k f(r, s), \quad k = 1, 2 \end{aligned}$$

Proof. We note that the graph of (V_1, \mathcal{F}_1) is identical with $V_1 \times \mathbb{R}^2$, so by [3, section 5], $J_1 = C^*(V_1, \mathcal{F}_1) \cong C_0(V_1) \rtimes_{\lambda} \mathbb{R}^2$. Similarly, we have

$$B_1 \cong C_0(W_1) \rtimes_{\lambda} \mathbb{R}^2,$$

$$J_2 \cong C_0(V_2) \rtimes_{\lambda} \mathbb{R}^2,$$

$$B_2 \cong C_0(W_2) \rtimes_{\lambda} \mathbb{R}^2,$$

From the equivariantly exact sequences in 2.1 and by [2, Lemma 1.1] we obtain the repeated extensions (γ_1) and (γ_2) .

Furthermore, the foliation (V_1, \mathcal{F}_1) can be derived from the submersion

$$p_1 : V_1 \approx \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^* \rightarrow \mathbb{R}^3 \cup \mathbb{R}^3$$

$$p_1(x, y, z, t, s) = (y, z, t, \text{signs}).$$

Hence, by a result of [3, p.562], we get $J_1 \cong C_0(\mathbb{R}^3 \cup \mathbb{R}^3) \otimes K$. The same argument shows that

$$J_2 \cong C_0(\mathbb{R}^2 \cup \mathbb{R}^2) \otimes K, \quad B_2 \cong C_0(\mathbb{R}_+) \otimes K.$$

□

3 COMPUTING THE INVARIANT SYSTEM OF $C^*(V, \mathcal{F})$

DEFINITION. The set of elements $\{\gamma_1, \gamma_2\}$ corresponding to the repeated extensions (γ_1) , (γ_2) in the Kasparov groups $\text{Ext}(B_i, J_i)$, $i = 1, 2$ is called the system of invariants of $C^*(V, \mathcal{F})$ and denoted by $\text{Index } C^*(V, \mathcal{F})$.

REMARK. $\text{Index } C^*(V, \mathcal{F})$ determines the so-called stable type of $C^*(V, \mathcal{F})$ in the set of all repeated extensions

$$\begin{aligned} 0 &\longrightarrow J_1 \longrightarrow E \longrightarrow B_1 \longrightarrow 0, \\ 0 &\longrightarrow J_2 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow 0. \end{aligned}$$

The main result of the paper is the following.

THEOREM 2. *Index $C^*(V, \mathcal{F}) = \{\gamma_1, \gamma_2\}$, where*

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ in the group } \text{Ext}(B_1, J_1) = \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2); \\ \gamma_2 &= (1, 1) \text{ in the group } \text{Ext}(B_2, J_2) = \text{Hom}(\mathbb{Z}, \mathbb{Z}^2). \end{aligned}$$

To prove this theorem, we need some lemmas as follows.

LEMMA 1. *Set $I_2 = C_0(\mathbb{R}^2 \times \mathbb{R}^*)$ and $A_2 = C_0((\mathbb{R}^2)^*)$*

The following diagram is commutative

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_j(I_2) & \longrightarrow & K_j(C_0(\mathbb{R}^3)^*) & \longrightarrow & K_j(A_2) \longrightarrow K_{j+1}(I_2) \longrightarrow \cdots \\ & & \downarrow \beta_1 & & \downarrow \beta_1 & & \downarrow \beta_1 \\ \cdots & \longrightarrow & K_{j+1}(C_0(V_2)) & \longrightarrow & K_{j+1}(C_0(W_1)) & \longrightarrow & K_{j+1}(C_0(W_2)) \longrightarrow K_j(C_0(V_2)) \longrightarrow \cdots \end{array}$$

where β_1 is the isomorphism defined in [13, Theorem 9.7] or in [2, corollary VI.3], $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. Let

$$\begin{aligned} k_2 : I_2 = C_0(\mathbb{R}^2 \times \mathbb{R}^*) &\rightarrow C_0((\mathbb{R}^3)^*) \\ v_2 : C_0((\mathbb{R}^3)^*) &\rightarrow A_2 = C_0((\mathbb{R}^2)^*) \end{aligned}$$

be the inclusion and restriction defined similarly as in 2.1.

One gets the exact sequence

$$0 \longrightarrow I_2 \xrightarrow{k_2} C_0((\mathbb{R}^3)^*) \xrightarrow{v_2} A_2 \longrightarrow 0$$

Note that

$$C_0(V_2) \cong C_0(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^*) \cong C_0(\mathbb{R}) \otimes I_2,$$

$$C_0(W_2) \cong C_0(\mathbb{R} \times (\mathbb{R}^2)^*) \cong C_0(\mathbb{R}) \otimes A_2,$$

$$C_0(W_1) \cong C_0(\mathbb{R} \times (\mathbb{R}^3)^*) \cong C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^3)^*.$$

The extension (2.1.2) thus can be identified to the following one

$$0 \longrightarrow C_0(\mathbb{R}) \otimes I_2 \xrightarrow{id \otimes k_2} C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^3)^* \xrightarrow{id \otimes v_2} C_0(\mathbb{R}) \otimes A_2 \longrightarrow 0.$$

Now, using [13, Theorem 9.7; Corollary 9.8] we obtain the assertion of Lemma 1. \square

LEMMA 2. Set $I_1 = C_0(\mathbb{R}^2 \times \mathbb{R}^*)$ and $A_1 = C(S^2)$

The following diagram is commutative

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_j(I_1) & \longrightarrow & K_j(C(S^3)) & \longrightarrow & K_j(A_1) \longrightarrow K_{j+1}(I_1) \longrightarrow \cdots \\ & & \downarrow \beta_2 & & \downarrow \beta_2 & & \downarrow \beta_2 \\ \cdots & \longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_j(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) \longrightarrow K_{j+1}(C_0(V_1)) \longrightarrow \cdots \end{array}$$

where β_2 is the Bott isomorphism, $j \in \mathbb{Z}/2\mathbb{Z}$.

Proof. The proof is similar to that of lemma 1, by using the exact sequence (2.1.1) and diffeomorphisms: $V \cong \mathbb{R} \times (\mathbb{R}^4)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^3$, $W_1 \cong \mathbb{R} \times (\mathbb{R}^3)^* \cong \mathbb{R} \times \mathbb{R}_+ \times S^2$. \square

Before computing the K-groups, we need the following notations. Let $u : \mathbb{R} \rightarrow S^1$ be the map

$$u(z) = e^{2\pi i(z/\sqrt{1+z^2})}, \quad z \in \mathbb{R}$$

Denote by u_+ (resp. u_-) the restriction of u on \mathbb{R}_+ (resp. \mathbb{R}_-). Note that the class $[u_+]$ (resp. $[u_-]$) is the canonical generator of $K_1(C_0(\mathbb{R}_+)) \cong \mathbb{Z}$ (resp. $K_1(C_0(\mathbb{R}_-)) \cong \mathbb{Z}$). Let us consider the matrix valued function $p : (\mathbb{R}^2)^* \cong S^1 \times \mathbb{R}_+ \rightarrow M_2(\mathbb{C})$ (resp. $\bar{p} : S^2 \cong D/S^1 \rightarrow M_2(\mathbb{C})$) defined by:

$$p(x; y) \text{ (resp. } \bar{p}(x, y)) = \frac{1}{2} \begin{pmatrix} 1 - \cos \pi \sqrt{x^2 + y^2} & \frac{x+iy}{\sqrt{x^2+y^2}} \sin \pi \sqrt{x^2 + y^2} \\ \frac{x-iy}{\sqrt{x^2+y^2}} \sin \pi \sqrt{x^2 + y^2} & 1 + \cos \pi \sqrt{x^2 + y^2} \end{pmatrix}.$$

Then p (resp. \bar{p}) is an idempotent of rank 1 for each $(x; y) \in (\mathbb{R}^2)^*$ (resp. $(x; y) \in D/S^1$). Let $[b] \in K_0(C_0(\mathbb{R}^2))$ be the Bott element, $[1]$ be the generator of $K_0(C(S^1)) \cong \mathbb{Z}$.

LEMMA 3 (See [15, p.234]).

- (i) $K_0(B_1) \cong \mathbb{Z}^2$, $K_1(B_1) = 0$,
- (ii) $K_0(J_2) \cong \mathbb{Z}^2$ is generated by $\varphi_0 \beta_1([b] \boxtimes [u_+])$ and $\varphi_0 \beta_1([b] \boxtimes [u_-])$; $K_1(J_2) = 0$,
- (iii) $K_0(B_2) \cong \mathbb{Z}$ is generated by $\varphi_0 \beta_1([1] \boxtimes [u_+])$; $K_1(B_2) \cong \mathbb{Z}$ is generated by $\varphi_1 \beta_1([p] - [\varepsilon_1])$,
where $\varphi_j, j \in \mathbb{Z}/2\mathbb{Z}$, is the Thom-Connes isomorphism (see [2]), β_1 is the isomorphism in Lemma 1, ε_1 is the constant matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and \boxtimes is the external tensor product (see, for example, [2, VI.2]).

LEMMA 4.

- (i) $K_0(C^*(V, \mathcal{F})) \cong \mathbb{Z}$, $K_1(C^*(V, \mathcal{F})) \cong \mathbb{Z}$,

- (ii) $K_0(J_1) = 0$; $K_1(J_1) \cong \mathbb{Z}^2$ is generated by $\varphi_1\beta_2([b] \boxtimes [u_+])$ and $\varphi_1\beta_2([b] \boxtimes [u_-])$,
 (iii) $K_1(B_1) = 0$; $K_0(B_1) \cong \mathbb{Z}^2$ is generated by $\varphi_0\beta_2[\bar{1}]$ and $\varphi_0\beta_2([\bar{p}] - [\varepsilon_1])$,
 where $\bar{1}$ is unit element in $C(S^2)$, φ_0 is the Thom-Connes isomorphism, β_2 is the Bott isomorphism.

Proof.

- (i) $K_i(C^*(V, \mathcal{F})) \cong K_i(C(S^3)) \cong \mathbb{Z}$, $i = 0, 1$.
 (ii) The proof is similar to (ii) of lemma 3.
 (iii) By [9, p.206], we have

$$K_0(C(S^2)) = \mathbb{Z}[\bar{1}] + \mathbb{Z}[q], \text{ where } q \in P_2(C(S^2)).$$

Otherwise, in [9, p.48,53,56]; [13, p.162], one has shown that the map

$$\dim : K_0(C(S^2)) \rightarrow \mathbb{Z}$$

is a surjective group homomorphism which satisfied $\dim[\bar{1}] = 1$, $\ker(\dim) = \mathbb{Z}$ and non-zero element $q \in P_2(C(S^2))$ in the kernel of the map \dim has the form $[q] = [\bar{p}] - [\varepsilon_1]$. Hence, the result is derived straight away because β_2 and φ_1 are isomorphisms. \square

Proof of theorem 2

1. Computation of (γ_1) . Recall that the extension (γ_1) in theorem 1 gives rise to a six-term exact sequence

$$\begin{array}{ccccc} 0 = K_0(J_1) & \longrightarrow & K_0(C^*(V, F)) & \longrightarrow & K_0(B_1) \\ & \delta_1 \uparrow & & & \downarrow \delta_0 \\ 0 = K_1(B_1) & \longleftarrow & K_1(C^*(V, F)) & \longleftarrow & K_1(J_1) \end{array}$$

By [11, Theorem 4.14], the isomorphisms

$$\text{Ext}(B_1, J_1) \cong \text{Hom}((K_0(B_1), K_1(J_1))) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$$

associates the invariant $\gamma_1 \in \text{Ext}(B_1, J_1)$ to the connecting map $\delta_0 : K_0(B_1) \rightarrow K_1(J_1)$. Since the Thom-Connes isomorphism commutes with K -theoretical exact sequence (see[14, Lemma 3.4.3]), we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_j(J_1) & \longrightarrow & K_j(C^*(V, F)) & \longrightarrow & K_j(B_1) \longrightarrow K_{j+1}(J_1) \longrightarrow \cdots \\ & & \uparrow \varphi_j & & \uparrow \varphi_j & & \uparrow \varphi_j \\ \cdots & \longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_j(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) \longrightarrow K_{j+1}(C_0(V_1)) \longrightarrow \cdots \\ & & & & & & \uparrow \varphi_{j+1} \end{array}$$

In view of Lemma 2, the following diagram is commutative

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_j(C_0(V_1)) & \longrightarrow & K_1(C_0(V)) & \longrightarrow & K_j(C_0(W_1)) \longrightarrow K_{j+1}(C_1(V_1)) \longrightarrow \cdots \\ & & \uparrow \beta_2 & & \uparrow \beta_2 & & \uparrow \beta_2 \\ \cdots & \longrightarrow & K_j(I_1) & \longrightarrow & K_j(C(S^3)) & \longrightarrow & K_j(A_1) \longrightarrow K_{j+1}(I_1) \longrightarrow \cdots \\ & & & & & & \uparrow \beta_2 \end{array}$$

Consequently, instead of computing $\delta_0 : K_0(B_1) \rightarrow K_1(J_1)$, it is sufficient to compute $\delta_0 : K_0(A_1) \rightarrow K_1(I_1)$. Thus, by the proof of Lemma 4, we have to define $\delta_0([p] - [\varepsilon_1]) = \delta_0([p])$ (because $\delta_0([\varepsilon_1]) = (0; 0)$ and $\delta_0([\bar{1}]) = (0; 0)$). By the usual definition (see [13, p.170]), for $[p] \in K_0(A_1)$, $\delta_0([p]) = [e^{2\pi i \tilde{p}}] \in K_1(I_1)$ where \tilde{p} is a preimage of p in (a matrix algebra over) $C(S^3)$, i.e. $v_1 \tilde{p} = p$.

We can choose $\tilde{p}(x, y, z) = \frac{z}{\sqrt{1+z^2}} \bar{p}(x, y)$, $(x, y, z) \in S^3$.

Let \tilde{p}_+ (resp. \tilde{p}_-) be the restriction of \tilde{p} on $\mathbb{R}^2 \times \mathbb{R}_+$ (resp. $\mathbb{R}^2 \times \mathbb{R}_-$). Then we have $\delta_0([p]) = [e^{2\pi i \tilde{p}}] = [e^{2\pi i \tilde{p}_+}] + [e^{2\pi i \tilde{p}_-}] \in K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_+)) \oplus K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_-)) = K_1(I_1)$

By [13, Section 4], for each function $f : \mathbb{R}_\pm \rightarrow Q_n \widetilde{C_0(\mathbb{R}^2)}$ such that $\lim_{x \rightarrow \pm 0} f(t) = \lim_{x \rightarrow \pm \infty} f(t)$, where $Q_n \widetilde{C_0(\mathbb{R}^2)} = \{a \in M_n \widetilde{C_0(\mathbb{R}^2)}, e^{2\pi i a} = Id\}$, the class $[f] \in K_1(C_0(\mathbb{R}^2) \otimes C_0(\mathbb{R}_\pm))$ can be determined by $[f] = W_f.[b] \boxtimes [u_\pm]$, where $W_f = \frac{1}{2\pi i} \int_{\mathbb{R}_\pm} Tr(f'(z)f^{-1}(z))dz$ is the winding number of f .

By simple computation, we get $\delta_0([p]) = [b] \boxtimes [u_+] + [b] \boxtimes [u_-]$. Thus $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z}^2)$.

2. Computation of (γ_2) . The extension (γ_2) gives rise to a six-term exact sequence

$$\begin{array}{ccccccc} K_0(J_2) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(B_2) & & \\ \uparrow \delta_1 & & & & \downarrow \delta_0 & & \\ K_1(B_2) & \longleftarrow & K_1(B_1) & \longleftarrow & K_1(J_2) = 0 & & \end{array}$$

By [11, Theorem 4.14], $\gamma_2 = \delta_1 \in \text{Hom}(K_1(B_2), K_0(J_2)) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}^2)$. Similarly to part 1, taking account of Lemma 1 and 3, we have the following commutative diagram ($j \in \mathbb{Z}/2\mathbb{Z}$)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & K_j(J_2) & \longrightarrow & K_j(B_1) & \longrightarrow & K_j(B_2) & \longrightarrow & K_{j+1}(J_2) & \longrightarrow & \cdots \\ & & \uparrow \varphi_j & & \uparrow \varphi_j & & \uparrow \varphi_j & & \uparrow \varphi_{j+1} & & \\ \cdots & \longrightarrow & K_j(C_0(V_2)) & \longrightarrow & K_j(C_0(W_1)) & \longrightarrow & K_j(C_0(W_2)) & \longrightarrow & K_{j+1}(C_0(V_2)) & \longrightarrow & \cdots \\ & & \uparrow \beta_1 & & \uparrow \beta_1 & & \uparrow \beta_1 & & \uparrow \beta_1 & & \\ \cdots & \longrightarrow & K_{j-1}(I_2) & \longrightarrow & K_{j-1}(C_0(\mathbb{R}^3)^*) & \longrightarrow & K_{j-1}(A_2) & \longrightarrow & K_j(I_2) & \longrightarrow & \cdots \end{array}$$

Thus we can compute $\delta_0 : K_0(A_2) \rightarrow K_1(I_2)$ instead of $\delta_1 : K_1(B_2) \rightarrow K_0(J_2)$. By the proof of Lemma 3, we have to define $\delta_0([p] - [\epsilon_1]) = \delta_0([p])$ (because $\delta_0([\epsilon_1]) = (0, 0)$). The same argument as above, we get $\delta_0([p]) = [b] \boxtimes [u_+] + [b] \boxtimes [u_-]$. Thus $\gamma_2 = (1, 1) \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}^2) \cong \mathbb{Z}^2$. The proof is completed. \square

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